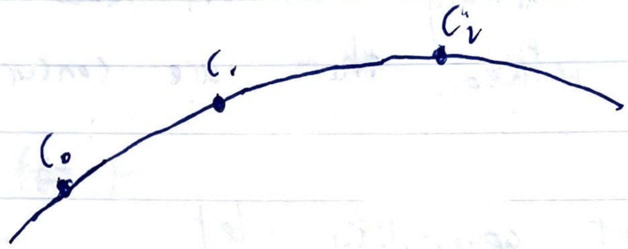
B-spline

7-9-2016



$c_0 = c_0(x_0, y_0) \in \mathbb{R}^2$

$t \in \mathbb{R}$

definition:

spline

$$q(t \mid c_0, c_1; t_0, t_1) = \frac{t_1 - t}{t_1 - t_0} c_0 + \frac{t - t_0}{t_1 - t_0} c_1 , \quad t \in [t_0, t_1]$$

$$q'(t \mid c_0, c_1; t_0, t_1) = \frac{c_1 - c_0}{t_1 - t_0}$$

$$|q'| = \frac{|c_1 - c_0|}{t_1 - t_0} \leftarrow$$ indicates how quickly as one travels from point c_0 to point c_1

three-point curve:

$$\begin{aligned} q_{0,2}(t) &= q(t \mid c_0, c_1, c_2; t_0, t_1, t_2) \\ &= \frac{t_2 - t}{t_2 - t_0} q_{0,1}(t) + \frac{t - t_0}{t_2 - t_0} q_{1,1}(t) \end{aligned}$$

$$q_{0,1}(t) = \frac{t_1 - t}{t_1 - t_0} c_0 + \frac{t - t_0}{t_1 - t_0} c_1 \qquad q_{1,1}(t) = \frac{t_2 - t}{t_2 - t_1} c_1 + \frac{t - t_1}{t_2 - t_1} c_2$$

note that the curve is not convex

$$q_{0,2}(t_0) = c_0$$

$$q_{0,2}(t_1) = \frac{t_2 - t_1}{t_2 - t_0} c_1 + \frac{t_1 - t_0}{t_2 - t_0} c_1 = c_1$$

$$q_{0,2}(t_2) = c_2$$

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thus, we can continue to 4 points,

$$q_{0,3}(t) = \frac{t_3 - t}{t_3 - t_0} q_{0,2}(t) + \frac{t - t_0}{t_3 - t_0} q_{1,2}(t)$$

Bézier curves

$$p_{1,1}(t) = p(t \mid C_0, C_1) = (1-t)C_0 + tC_1 \qquad t \in [0,1]$$

$$p_{2,1}(t) = p(t \mid C_1, C_2) = (1-t)C_1 + tC_2$$

the curve is convex !!!

$$p_{2,2}(t) = p(t \mid C_0, C_1, C_2) = (1-t)p_{1,1}(t) + t\,p_{2,1}(t)$$

$$= (1-t)^2 C_0 + 2t(1-t)C_1 + t^2 C_2$$

accordingly, $p_{3,3}(t) = (1-t)p_{2,2}(t) + t\,p_{3,2}(t)$

in general, $p_{d,d}(t) = (1-t)p_{d-1,d-1}(t) + t\,p_{d,d-1}(t)$

$$= b_{0,d}(t)C_0 + \dots + b_{d,d}(t)C_d$$

where $b_{i,d}(t) = \binom{d}{i} t^i (1-t)^{d-i}$

actually, t can belong to any interval set $[a,b]$

$$p'_{d,d}(t) = \frac{d}{dt}\Big[(1-t)^d C_0 + \binom{d}{1} t(1-t)^{d-1} C_1 + \dots$$

$$+ \binom{d}{1} t^{d-1}(1-t)C_{d-1} + t^d C_d\Big]$$

$$= -d(1-t)^{d-1}C_0 + \binom{d}{1}(1-t)^{d-1}C_1 - (d-1)\binom{d}{1} t(1-t)^{d-2}C_1$$

$$+ \dots + (d-1)\binom{d}{1} t^{d-2}(1-t)C_{d-1} - \binom{d}{1} t^{d-1} C_{d-1}$$

$$+ d\,t^{d-1} C_d$$

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$\therefore \quad p'_{d,d}(0) = d(c_1 - c_0)$

$\quad p'_{d,d}(1) = d(c_d - c_{d-1})$

<u>composite Bézier curves</u>

"glue" two Bézier curves together, $(c_0^i, \cdots, c_d^i)_{i=1}^n$

continuity condition : C^0, $c_d^{i-1} = c_0^i$

C^1, $c_d^{i-1} - c_{d-1}^{i-1} = c_1^i - c_0^i$

C^2, $\boxed{?}$ $c_d^{i-1} - 2c_{d-1}^{i-1} + c_{d-2}^{i-1}$

$= c_0^i - 2c_1^i + c_2^i$

$\vdots$

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if we have 2 points c_1 and c_2, and $t \in [t_2, t_3]$

let $p(t \mid c_1, c_2; t_2, t_3) = \dfrac{t_3 - t}{t_3 - t_2} c_1 + \dfrac{t - t_2}{t_3 - t_2} c_2$

if $t_2 = 0$, $t_3 = 1$, we get back to Bézier curve.

define

$$f(t) := \begin{cases} p(t \mid c_1, c_2; t_2, t_3) & t \in [t_2, t_3] \\ p(t \mid c_2, c_3; t_3, t_4) & t \in [t_3, t_4] \\ \vdots & \vdots \\ p(t \mid c_{n-1}, c_n \mid t_n, t_{n+1}) & t \in [t_n, t_{n+1}] \end{cases}$$

B-spline

define

$$B_{i,0}(t) = \begin{cases} 1 & t_i \le t < t_{i+1} \\ 0 & \text{else} \end{cases}$$

$$\therefore f(t) = \sum_{i=2}^{n} p_{i,1}(t) B_{i,0}(t)$$

where $p_{i,1}(t) = p(t \mid c_{i-1}, c_i; t_i, t_{i+1})$

this is linear spline curve.

Interplating polynomial spline function is not convex, while Bézier curve is convex but hard to visualise when the degree is high,

$$p = \sum_{i=0}^{n+j} p_i^j B_{i,d-j}$$

$$p_i^j = \frac{t - t_i}{t_{i+d+1-j} - t_i} p_i^{j-1} + \frac{t_{i+d+1-j} - t}{t_{i+d+1-j} - t_i} p_{i-1}^{j-1}$$

$$p = \sum_{i=0}^{n} p_i B_{i,d}$$

$n \geq d$

$t \in [t_d, t_{n+1}]$

$\vec{t} = (t_0, \ldots, t_{n+\blacksquare+1})$

Ex.) $n = d$

$d = 1$

(t_0, t_1, t_2, t_3)

redundant

so, first, as an example,

define quadratic spline curve for c_1, c_2, c_3, with $t \in [t_2, t_3, t_4, t_5]$

$$\left.\begin{array}{l} p(t \mid c_1, c_2; t_2, t_4) \\ p(t \mid c_2, c_3; t_3, t_5) \end{array}\right\} \to p(t \mid c_1, c_2, c_3; \cancel{t_3, t_4)}\ t_2, t_3, t_4, t_5)$$

$$= \frac{t_4 - t}{t_4 - t_3} p(t \mid c_1, c_2 \cancel{; t_2, t_4)}\ ; t_2, t_4) + \frac{t - t_3}{t_4 - t_3} p(t \mid c_2, c_3; t_3, t_5)$$

note that $t \in [t_3, t_4]$ for the curve to satisfy convex condition.

if $t_2 = t_3 = 0$, $t_4 = t_5 = 1$, we get back to Bézier curve.

B-spline

$$f(t) = \begin{cases} p(t \mid c_1, c_2, c_3; t_2, t_3, t_4, t_5) & t \in [t_3, t_4] \\ p(t \mid c_2, c_3, c_4; t_3, t_4, t_5, t_6) & t \in [t_4, t_5] \\ \vdots & \vdots \\ p(t \mid c_{n-2}, c_{n-1}, c_n; t_{n-1}, t_n, t_{n+1}, t_{n+2}) & t \in [t_n, t_{n+1}] \end{cases}$$

$$f(t) = \sum_{i=3}^{n} p_{i,2}(t) B_{i,0}(t)$$

where $p_{i,2}(t) = p(t \mid c_{i-2}, c_{i-1}, c_i; t_{i-1}, t_i, t_{i+1}, t_{i+2})$

accordingly, cubic spline:

$$p_{i,3}(t) = p(t \mid c_{i-3}, c_{i-2}, c_{i-1}, c_i; t_{i-2}, t_{i-1}, t_i, t_{i+1}, t_{i+2}, t_{i+3})$$

$$= p(t \mid c_{i-3}, c_{i-2}, c_{i-1}; t_{i-2}, t_{i-1}, t_{i+1}, t_{i+2}) \cdot \frac{t_{i+1} - t}{t_{i+1} - t_i} +$$

$$p(t \mid c_{i-2}, c_{i-1}, c_i; t_{i-1}, t_i, t_{i+2}, t_{i+3}) \cdot \frac{t - t_i}{t_{i+1} - t_i}$$

$t \in [t_i, t_{i+1}]$

in general,

let $p^s_{i,k}(t) = p(t \mid c_{i-k}, \dots, c_i; t_{i-k+1}, \dots, t_i, t_{i+s}, \dots t_{i+k+s-1})$

note that $s + k = d + 1$, d = degree

$\therefore$ for example, $p^1_{i,3}(t) = \frac{t_{i+1} - t}{t_{i+1} - t_i} p^2_{i-1,2}(t) + \frac{t - t_i}{t_{i+1} - t_i} p^2_{i,2}(t)$

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$$\therefore\ p_{i,d}(t) = \frac{t_{i+1}-t}{t_{i+1}-t_i} p_{i-1,d-1}(t) + \frac{t-t_i}{t_{i+1}-t_i} p_{i,d-1}(t)$$

$$p_{i,d-1}(t) = \frac{t_{i+2}-t}{t_{i+2}-t_i} p_{i-1,d-2}(t) + \frac{t-t_i}{t_{i+2}-t_i} p_{i,d-2}(t)$$

$$\vdots$$

$$p_{i,1}(t) = \frac{t_{i+d}-t}{t_{i+d}-t_i} c_{i-1} - \frac{t-t_i}{t_{i+d}-t_i} c_i$$

n control points

n+d+1 knots

let $c = (c_i)_{i=1}^{n}$, $t_a = (t_i)_{i=2}^{n+d}$, $t \in [t_{d+1}, t_{n+1}]$

$$\boxed{f(t) = \sum_{i=d+1}^{n} p_{i,d}(t) B_{i,0}(t)}$$

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$$f(t) = \sum_{i=d+1}^{n} p_{i,d}(t) B_{i,0}(t)$$

$$= \sum_{i=d+1}^{n} \left(\frac{t_{i+1}-t}{t_{i+1}-t_i} p_{i-1,d-1}(t) + \frac{t-t_i}{t_{i+1}-t_i} p_{i,d-1}(t) \right) B_{i,0}(t)$$

$$= \sum_{i=d+1}^{n-1} \left(\frac{t-t_i}{t_{i+1}-t_i} B_{i,0}(t) + \frac{t_{i+2}-t}{t_{i+2}-t_{i+1}} B_{i+1,0}(t) \right) p_{i,d-1}(t)$$

$$+ \frac{t_{d+2}-t}{t_{d+2}-t_{d+1}} B_{d+1,0}(t)\, p_{d,d-1}(t) + \frac{t-t_n}{t_{n+1}-t_n} B_{n,0}(t)\, p_{n,d-1}(t)$$

since $t \in [t_{d+1}, t_{n+1}]$, $B_{d,0}(t) = 0 = B_{n+1,0}(t)$

B-spline

$$\therefore f(t) = \sum_{i=d+1}^{n-1} \left(\frac{t-t_i}{t_{i+1}-t_i} B_{i,0}(t) + \frac{t_{i+2}-t}{t_{i+2}-t_{i+1}} B_{i+1,0}(t) \right) p_{i,d-1}(t)$$

$$+ \left(\frac{t-t_d}{t_{d+1}-t_d} B_{d,0}(t) + \frac{t_{d+2}-t}{t_{d+2}-t_{d+1}} B_{d+1,0}(t) \right) p_{d,d-1}(t)$$

$$+ \left(\frac{t-t_n}{t_{n+1}-t_n} B_{n,0}(t) + \frac{t_{n+2}-t}{t_{n+2}-t_{n+1}} B_{n+1,0}(t) \right) p_{n,d-1}(t)$$

$$\boxed{f(t) = \sum_{i=d}^{n} p_{i,d-1}(t) B_{i,1}(t)}$$

$$B_{i,1}(t) = \frac{t-t_i}{t_{i+1}-t_i} B_{i,0}(t) + \frac{t_{i+2}-t}{t_{i+2}-t_{i+1}} B_{i+1,0}(t), \quad t \in [t_i, t_{i+2}]$$

accordingly,

$$\boxed{\begin{aligned} B_{i,r}(t) &= \frac{t-t_i}{t_{i+r}-t_i} B_{i,r-1}(t) + \frac{t_{i+r+1}-t}{t_{i+r+1}-t_{i+1}} B_{i+1,r-1}(t) \qquad t \in [t_i, t_{i+r+1}] \\ B_{i,r}(t) &= 0 \begin{cases} \text{if} & t < t_i, \text{ or } t \geq t_{i+r+1} \\ \text{if} & t_i = t_{i+r+1} \end{cases} \end{aligned}}$$

inductively, let it be true $f(t) = \sum_{i=d-r+1}^{n} p_{i,d-r+1}(t) B_{i,r-1}(t)$

$r > 1$

$$f(t) = \sum_{i=d-r+2}^{n} \left(\frac{t_{i+r}-t}{t_{i+r}-t_i} p_{i-1,d-r}(t) + \frac{t-t_i}{t_{i+r}-t_i} p_{i,d-r}(t) \right) B_{i,r-1}(t)$$

$$= \sum_{i=d-r+2}^{n} \left(\frac{t-t_i}{t_{i+r}-t_i} B_{i,r-1}(t) + \frac{t_{i+r+1}-t}{t_{i+r+1}-t_{i+1}} B_{i+1,r-1}(t) \right) p_{i,d-r}(t)$$

remember that $t \in [t_{d+1}, t_{n+1}]$

$$+ \frac{t_{d+2}-t}{t_{d+2}-t_{d-r+2}} B_{d-r+2,r-1}(t) p_{d-r+1,d-r}(t)$$

$$+ \frac{t-t_{d-r+1}}{t_{d+1}-t_{d-r+1}} B_{d-r+1,r-1}(t) p_{d-r+1,r-1}(t)$$

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$$+\frac{t-t_n}{t_{n+r}-t_n}B_{n,r-1}(t)\,p_{n,d-r}(t)$$

$$+\frac{t_{n+r+1}-t}{t_{n+r+1}-t_{n+1}}B_{n+1,r-1}(t)\,p_{n,d-r}(t)$$

$$\therefore f(t)=\sum_{i=d-r+1}^{n}\left(\frac{t-t_i}{t_{i+r}-t_i}B_{i,r-1}(t)+\frac{t_{i+r+1}-t}{t_{i+r+1}-t_{i+1}}B_{i+1,r-1}(t)\right)p_{i,d-r}(t)$$

B-spline!!!

$$\boxed{f(t)=\sum_{i=d-r+1}^{n}p_{i,d-r}(t)\,B_{i,r}(t)}$$

$$f(t)=\sum_{i=1}^{n}p_{i,0}(t)\,B_{i,d}(t)$$

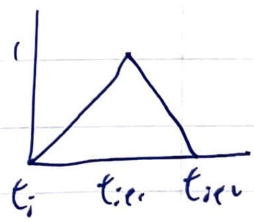
$$=\sum_{i=1}^{n}c_i\,B_{i,d}(t)$$

Properties of splines and B-splines

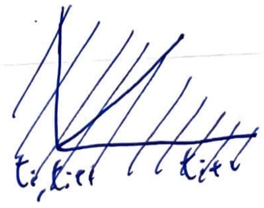
define $$B_{i,d,\vec{t}}(x)=\frac{x-t_i}{t_{i+d}-t_i}B_{i,d-1,\vec{t}}(x)+\frac{t_{i+d+1}-x}{t_{i+d+1}-t_{i+1}}B_{i+1,d-1,\vec{t}}(x)$$

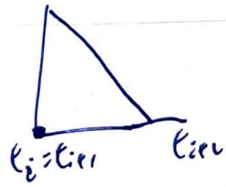
$$B_{i,0,\vec{t}}(x)=\begin{cases}1 & t_i\le x<t_{i+1}\\ 0 & \text{else}\end{cases}$$

$$B_{i,1,\vec{t}}(x)=\begin{cases}\dfrac{x-t_i}{t_{i+1}-t_i} & t_i\le x<t_{i+1}\\ \dfrac{t_{i+2}-x}{t_{i+2}-t_{i+1}} & t_{i+1}\le x<t_{i+2}\\ 0 & \text{else}\end{cases}$$



if $t_i=t_{i+1}$





B-spline

$$B_d^T(x) = (B_{\mu-d,d}\ \ B_{\mu-d+1,d}\ \cdots\ B_{\mu,d}) = R_1(x)R_2(x)\cdots R_d(x)$$

$$\therefore f(x) = R_1(x)R_2(x)\cdots R_d(x)\,c_d$$

where $c_d = (c_{\mu-d}, \cdots, c_\mu)$

Linear Independence

define dual polynomial:

$$\rho_{j,0}(y) = 1$$

$$\rho_{j,d}(y) = (y-t_{j+1})\cdots(y-t_{j+d})$$

$$B_d = (B_{\mu-d} \cdots B_\mu)^T \Leftrightarrow \rho_d(y) = (\rho_{\mu-d,d}(y) \cdots \rho_{\mu,d}(y))^T$$

$$\rho_{d-1}(y) = (\rho_{\mu-d+1,d-1}(y) \cdots \rho_{\mu,d-1}(y))^T$$

$$\boxed{R_d(x)\rho_d(y) = (y-x)\rho_{d-1}(y)}$$

proof:

$$\frac{(x-t_j)\rho_{j,d}(y) + (t_{j+d}-x)\rho_{j-1,d}(y)}{t_{j+d}-t_j} = (y-x)\rho_{j,d-1}(y)$$

because $\rho_{j,d}(y) = (y-t_{j+d})\rho_{j,d-1}(y)$

$\rho_{j-1,d}(y) = (y-t_j)\rho_{j,d-1}(y)$

$$\therefore R_1(x_1)R_2(x_2)\cdots R_d(x_d)\rho_d(y) = (y-x_1)(y-x_2)\cdots(y-x_d)$$

B-spline

$$\boxed{R_{d-1}(x)R_d(z) = R_{d-1}(x)R_d(x)}$$

$$\begin{aligned} R_{d-1}(x)R_d(z)P_d(y) &= (y-x)(y-z)P_{d-2}(y) \\ &= (y-z)(y-x)P_{d-2}(y) \\ &= R_{d-1}(z)R_d(x)P_{d-2}(y) \end{aligned}$$

let $B = R_{d-1}(x)R_d(z) - R_{d-1}(z)R_d(x)$

$B P_d(y) = 0$

arbitrary a, thus $a^T B P_d(y) = 0$

since $P_d(y)$ are linearly independent, $a^T B = 0$

$\therefore B = 0$

this is also valid for $x \in [t_{d+1}, t_{n+1})$

$(y-x)^d = \sum_{j=1}^{n+1} B_{j,d}(x) P_{j,d}(y)$

because it is independent of μ

$(y-x)^d = \sum_{j=1}^{d+1} B\,P$

$(y-x)^d = \sum_{j=2}^{d+2} B\,P$

$\Downarrow$

$(y-x)^d = \sum_{j=1}^{d+2} B\,P$

if $x \in [t_{d+2}, t_{d+3})$

$B_{j=1} = 0$

$$\boxed{\begin{aligned} &\text{let } B_d(x) = R_1(x)\cdots R_d(x) \\ &(y-x)^d = B_d^T(x) P_d(y) = \sum_{j=\mu-d}^{\mu} B_{j,d}(x) P_{j,d}(y) \\ &\qquad\qquad x \in [t_\mu, t_{\mu+1}) \end{aligned}}$$

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since $(y-x)^d = \sum_{j=\mu-d}^{\mu} B_{j,d}(x) P_{j,d}(y)$

$$\left.\frac{d^{d-r}}{dy^{d-r}}(y-x)^d\right|_{y=0} = \sum_{j=\mu-d}^{\mu} B_{j,d}(x) \left.\frac{d^{d-r}}{dy^{d-r}}P_{j,d}(y)\right|_{y=0}$$

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$$\frac{\partial^{d-r}}{\partial y^{d-r}}(y-x)^d\Big|_{y=0} = \cancel{\frac{d!}{(d-d+r)!}}\,(y-x)^r\Big|_{y=0}$$

$$= \frac{d!}{r!}(-1)^r x^r$$

$$P_{j,d}(y) = y^d - t^*_{j,d}\,y^{d-1} + t^{**}_{j,d}\,y^{d-2} + \dots$$

where $t^*_{j,d} = t_{j+1} + t_{j+2} + \dots + t_{j+d}$

$$t^{**}_{j,d} = \sum_{\substack{m,n=1 \\ m\neq n}}^{d} t_{j+m}t_{j+n}$$

$$\therefore \frac{\partial^{d-r}}{\partial y^{d-r}}P_{j,d}(y)\Big|_{y=0} = \cancel{\frac{d!}{r!}}\,y^r - \cancel{\frac{(d-1)!}{(r-1)!}}\,t^*_{j,d}\,y^{r-1} + \dots (d-r)!\,t^{(*)r}_{j,d}\Big|_{y=0}$$

$$= (d-r)!\,t^{(*)r}_{j,d}$$

$\therefore$ $r=0$, $\frac{\partial^d}{\partial y^d}P_{j,d}(y)\Big|_{y=0} = d!$

$r=1$, $\frac{\partial^{d-1}}{\partial y^{d-1}}P_{j,d}(y)\Big|_{y=0} = (d-1)!\,t^*_{j,d}$

$\vdots$

Hence,
$$d! = \sum_{j=\mu-d}^{\mu} B_{j,d}(x)\cdot d!$$

$$d!(-1)x = \sum_{j=\mu-d}^{\mu} B_{j,d}(x)\,(d-1)!\,t^*_{j,d}(-1)$$

$$\frac{d!}{2!}x^2 = \sum_{j=\mu-d}^{\mu} B_{j,d}(x)\,\frac{(d-2)!}{1!}\,t^{**}_{j,d}$$

$\dots$

B-spline

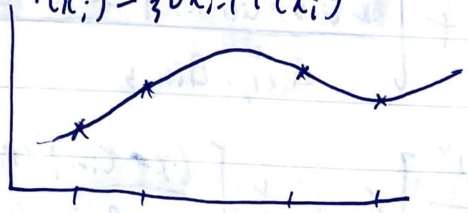
Hermite cubic polynomial

$$Hf = \sum_{i=1}^{2m} c_i B_{i,3} \qquad t = (t_i)_{i=1}^{2m+4}$$

$$c_{2i} = f(x_i) + \frac{1}{3}\Delta x_i f'(x_i)$$

$$c_{2i-1} = f(x_i) - \frac{1}{3}\Delta x_{i-1} f'(x_i)$$

$$= (x_1, x_1, x_1, x_1, x_2, x_2, \cdots, x_{m-1}, x_{m-1}, x_m, x_m, x_m, x_m)$$



$t \in [x_i, x_{i+1})$ $\quad x_i \neq x_{i+1}$ $\quad , t_i = x_i, \ t_{i+1} = x_{i+1},$
$t_{i+2} = x_{i+1}, \ t_{i+3} = t_{i+2}$

$$\begin{pmatrix} \frac{t_{i+1}-x}{\Delta_{i+1,1}} & \frac{x-t_i}{\Delta_{i+1,1}} \end{pmatrix} \begin{pmatrix} \frac{t_{i+1}-x}{\Delta_{i+1,2}} & \frac{x-t_{i-1}}{\Delta_{i+1,2}} & 0 \\ 0 & \frac{t_{i+2}-x}{\Delta_{i+2,2}} & \frac{x-t_i}{\Delta_{i+2,2}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{t_{i+1}-x}{\Delta_{i+1,3}} & \frac{x-t_{i-2}}{\Delta_{i+1,3}} & 0 & 0 \\ 0 & \frac{t_{i+2}-x}{\Delta_{i+2,3}} & \frac{x-t_{i-1}}{\Delta_{i+2,3}} & 0 \\ 0 & 0 & \frac{t_{i+3}-x}{\Delta_{i+3,3}} & \frac{x-t_i}{\Delta_{i+3,3}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{t_{i+1}-x}{\Delta_{i+1,1}} & \frac{x-t_i}{\Delta_{i+1,1}} \end{pmatrix} \begin{pmatrix} \frac{(t_{i+1}-x)^2}{\Delta_{i+1,2}\Delta_{i+1,3}} & \frac{(x-t_{i-2})(t_{i+1}-x)+(x-t_{i-1})(t_{i+2}-x)}{\Delta_{i+1,2}\Delta_{i+1,3}} & \frac{(x-t_{i-1})^2}{\Delta_{i+1,2}\Delta_{i+2,3}} & 0 \\ 0 & \frac{(t_{i+2}-x)^2}{\Delta_{i+2,2}\Delta_{i+2,3}} & \frac{(x-t_{i-1})(t_{i+2}-x)+(x-t_i)(t_{i+3}-x)}{\Delta_{i+2,2}\Delta_{i+2,3}} & \frac{(x-t_i)^2}{\Delta_{i+2,2}\Delta_{i+2,3}} \end{pmatrix}$$

B-spline

$x_{i-1}, x_i, x_{i+1}, x_{i+2}$

$t_{i-1}, t_i, t_{i+1}, t_{i+2}$

$x = x_i$, $\quad p(x_i) = \frac{\Delta_{i+1,1}}{\Delta_{i+1,3}} C_{i-3} + \frac{\Delta_{i,2}}{\Delta_{i+1,3}} C_{i-2} + \frac{\Delta_{i+1,1}}{\Delta_{i+2,3}} C_{i-1}$

$$p(x) = \frac{(t_{i+1}-x)^3}{\Delta_{i+1,1}^2 \Delta_{i+1,3}} C_{i-3} + \left[\frac{(x-t_{i-2})(t_{i+1}-x)^2 + (x-t_{i-1})(t_{i+1}-x)(t_{i+2}-x)}{\Delta_{i+1,1}^2 \Delta_{i+1,3}} + \frac{(x-t_i)(t_{i+2}-x)^2}{\Delta_{i+1,1}^2 \Delta_{i+2,3}} \right] C_{i-2} + \left[\frac{(x-t_{i-1})^2(t_{i+1}-x)}{\Delta_{i+1,1}^2 \Delta_{i+1,3}} + \frac{(x-t_{i-1})(x-t_i)(t_{i+2}-x) + (x-t_i)^2(t_{i+3}-x)}{\Delta_{i+1,1}^2 \Delta_{i+2,3}} \right] C_{i-1} + \frac{(x-t_i)^3}{\Delta_{i+1,1}^2 \Delta_{i+2,3}} C_i$$

$$p'(x) = \frac{-3(t_{i+1}-x)^2}{\Delta_{i+1,1}^2 \Delta_{i+1,3}} C_{i-3} + \left[\frac{(t_{i+1}-x)^2 - 2(x-t_{i-2})(t_{i+1}-x) + (t_{i+1}-x)(t_{i+2}-x)}{\Delta_{i+1,1}^2 \Delta_{i+1,3}} + \frac{(t_{i+1}-x)(t_{i+2}-x) + (t_{i+1}-x)(t_{i+1}-x)}{\Delta_{i+1,1}^2 \Delta_{i+1,3}} + \frac{(t_{i+2}-x)^2 - 2(x-t_i)(t_{i+2}-x)}{\Delta_{i+1,1}^2 \Delta_{i+2,3}} \right] C_{i-2}$$

$$+ \left[\frac{(x-t_i)(t_{i+2}-x) + (x-t_{i-1})(t_{i+2}-x) - (x-t_{i-1})(x-t_i) + 2(x-t_i)(t_{i+3}-x)}{\Delta_{i+1,1}^2 \Delta_{i+2,3}} + \frac{-(x-t_i)^2}{\Delta_{i+1,1}^2 \Delta_{i+2,3}} + \frac{2(x-t_{i-1})(t_{i+1}-x) - (x-t_{i-1})^2}{\Delta_{i+1,1}^2 \Delta_{i+1,3}} \right] C_{i-1}$$

$$+ \frac{3(x-t_i)^2}{\Delta_{i+1,1}^2 \Delta_{i+2,3}} C_i$$

$$p'(x_i) = \frac{-3}{\Delta_{i+1,3}} C_{i-3} + \left[\frac{\Delta_{i+1,1} - \Delta_{i,2}}{\Delta_{i+1,1} \Delta_{i+1,3}} + \frac{1}{\Delta_{i+2,3}} \right] C_{i-2}$$

B-spline

$t \in [x_{i+1}, x_{i+2})$

$$p(x_{i+1}) = \frac{\Delta_{i+3,1}}{\Delta_{i+3,3}} c_{i-1} + \frac{\Delta_{i+2,1}}{\Delta_{i+3,3}} c_i + \frac{\Delta_{i+3,1}}{\Delta_{i+4,3}} c_{i+1}$$

$$p'(x_{i+1}) = \frac{-3}{\Delta_{i+3,3}} c_{i-1} + \left[\frac{\Delta_{i+3,1} - \Delta_{i+2,2}}{\Delta_{i+3,1}\,\Delta_{i+3,3}} + \frac{1}{\Delta_{i+4,3}} \right] c_i$$

with $p(x_i)$, $p'(x_i)$, $p'(x_{i+1})$, solve for $c_{i-3}, c_{i-2}, c_{i-1}$

Least Square B-Spline

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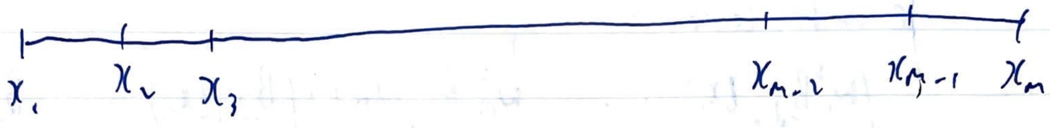
$$\delta = \sum_{r=1}^{m}\left[w_r\left(y_r - \sum_{i=-d}^{g} c_i B_{i,d}(x_r)\right)\right]^2$$



$$a \le x_i \le b$$

$$\delta = (y - Ec)^T (y - Ec) = \|y - Ec\|^2$$

$$y = \begin{pmatrix} w_1 y_1 \\ \vdots \\ w_m y_m \end{pmatrix} \qquad E = \begin{pmatrix} w_1 B_{-d,d}(x_1) & w_1 B_{-d+1,d}(x_1) & \cdots & w_1 B_{g,d}(x_1) \\ w_2 B_{-d,d}(x_2) & & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ w_m B_{-d,d}(x_m) & & \cdots & w_m B_{g,d}(x_m) \end{pmatrix}$$

$$c = \begin{pmatrix} c_{-d} \\ \vdots \\ c_g \end{pmatrix}$$

$$\frac{\partial \delta}{\partial c} = 0 \implies 0 = 2\sum_{j=-d}^{g}\sum_{r=1}^{m}\left[w_r\left(y_r - \sum_{i=-d}^{g} c_i B_{i,d}(x_r)\right)\right] w_r B_{j,d}(x_r)$$

$$\therefore 0 = \sum_{j=-d}^{g}\sum_{r=1}^{m} w_r^2 y_r B_{j,d}(x_r) - \sum_{j=-d}^{g}\sum_{r=1}^{m}\sum_{i=-d}^{g} w_r^2 c_i B_{i,d}(x_r) B_{j,d}(x_r)$$

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Least Square B-spline

first term:
$$\begin{pmatrix} w_1^2 B_{-d,d}(x_1) & \cdots & w_m^2 B_{-d,d}(x_m) \\ \vdots & \ddots & \vdots \\ w_1^2 B_{g,d}(x_1) & \cdots & w_m^2 B_{g,d}(x_m) \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

Second term:
$$\begin{pmatrix} w_1^2 B_{-d,d}(x_1) & \cdots & w_m^2 B_{-d,d}(x_m) \\ \vdots & \ddots & \vdots \\ w_1^2 B_{g,d}(x_1) & \cdots & w_m^2 B_{g,d}(x_m) \end{pmatrix} \begin{pmatrix} B_{-d,d}(x_1) & \cdots & B_{g,d}(x_1) \\ \vdots & \ddots & \vdots \\ B_{-d,d}(x_m) & \cdots & B_{g,d}(x_m) \end{pmatrix} \begin{pmatrix} c_{-d} \\ \vdots \\ c_g \end{pmatrix}$$

$$= \begin{pmatrix} \langle B_{-d,d}, B_{-d,d} \rangle & \cdots & \langle B_{-d,d}, B_{g,d} \rangle \\ \vdots & \ddots & \vdots \\ \langle B_{g,d}, B_{-d,d} \rangle & \cdots & \langle B_{g,d}, B_{g,d} \rangle \end{pmatrix} \begin{pmatrix} c_{-d} \\ \vdots \\ c_g \end{pmatrix}$$

$$\langle B_{i,d}, B_{j,d} \rangle = \sum_{r=1}^{m} w_r^2 B_{i,d}(x_r) B_{j,d}(x_r)$$

$\therefore Ac = r$

note that: $\langle B_{i,d}, B_{j,d} \rangle = 0$ if $|i-j| > d$

A is a positive-definite, symmetric matrix

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$$\therefore \begin{pmatrix} A_{1,1} & \cdots & A_{1,d+1} & 0 & & & \\ A_{2,1} & A_{2,2} & \cdots & A_{2,d+2} & 0 & & \\ & & \ddots & & & & \\ 0 & A_{i,i-d} & \cdots & A_{i,i} & \cdots & A_{i,i+d} & 0 \\ & 0 & & & \ddots & & \\ & & 0 & A_{g,g-d} & \cdots & & A_{g,g} \end{pmatrix}$$

periodic boundary condition:

$$t_{-i} = t_{g+1-i} - b + a$$
$$t_{g+1+i} = t_i + b - a \qquad 0 \le i \le d$$

$t_i \le x \le t_{i+1}$

prove that $B_{i,k} = (t_{i+k+1} - t_i) \sum_{j=0}^{k+1} \frac{(t_{i+j} - x)_+^k}{\prod_{\substack{l=0 \\ l \ne j}}^{k+1} (t_{i+j} - t_{i+l})}$

$(t_{i+j} - x)_+^k$

$x \le t_{i+j}$

$$B_{i,0} = 1 = R.H.S. = (t_1 - t_0)\left(\frac{1}{t_1 - t_0}\right)$$

$$B_{i,1} = \frac{x - t_i}{t_{i+1} - t_i} B_{i,0} + \frac{t_{i+2} - x}{t_{i+2} - t_{i+1}} B_{i+1,0}$$

$$RHS = \frac{(t_{i+2} - x)_+}{t_{i+2} - t_{i+1}} + \frac{(t_{i+1} - x)_+ (t_{i+2} - t_i)}{(t_{i+1} - t_i)(t_{i+1} - t_{i+2})}$$

$$\cancel{B_{i,1} = \frac{x - t_i}{t_{i+1} - t_i} B_{i,0} -}$$

$$= \frac{t_{i+2} - x}{t_{i+2} - t_{i+1}} B_{i+1,0} + \frac{(t_{i+1} - x)(t_{i+2} - t_i)}{(t_{i+1} - t_i)(t_{i+1} - t_{i+2})} B_{i,0}$$
$$+ \frac{t_{i+2} - x}{t_{i+2} - t_{i+1}} B_{i,0}$$

$$= \frac{t_{i+2} - x}{t_{i+2} - t_{i+1}} B_{i+1,0} + \left[\frac{(t_{i+1} - x)(t_i - t_{i+2}) + (t_{i+2} - x)(t_{i+1} - t_i)}{(t_{i+2} - t_{i+1})(t_{i+1} - t_i)}\right] B_{i,0}$$

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~~Cholesky Decomposition~~

$$\text{R.H.S.} = \frac{t_{i+2}-x}{t_{i+2}-t_{i+1}} B_{i+1,0} + \left[\frac{(t_{i+1}-t_{i+2})t_i + x(t_{i+2}-t_{i+1})}{(t_{i+2}-t_{i+1})(t_{i+1}-t_i)}\right] B_{i,0}$$

$$= \frac{x-t_i}{t_{i+1}-t_i} B_{i,0} + \frac{t_{i+2}-x}{t_{i+2}-t_{i+1}} B_{i+1,0}$$

$$= B_{i,1}$$

when $j=n$, by induction, show that $n+1$ is true

$$B_{i,n+1} = \frac{x-t_i}{t_{i+n+1}-t_i} B_{i,n} + \frac{t_{i+n+2}-x}{t_{i+n+2}-t_{i+1}} B_{i+1,n}$$

$$= \frac{x-t_i}{t_{i+n+1}-t_i}(t_{i+n+1}-t_i)\sum_{j=0}^{n+1}\frac{(t_{i+j}-x)_+^n}{\prod_{\substack{l=0\\l\neq j}}^{n+1}(t_{i+j}-t_{i+l})}$$

$$+ \frac{t_{i+n+2}-x}{t_{i+n+2}-t_{i+1}}(t_{i+n+2}-t_{i+1})\sum_{j=0}^{n+1}\frac{(t_{i+1+j}-x)_+^n}{\prod_{\substack{l=0\\l\neq j}}^{n+1}(t_{i+1+j}-t_{i+1+l})}$$

consider the terms with same $(t_{i+j+1}-x)_+^n$ in both sums

$$\frac{x-t_i}{t_{i+n+1}-t_i}(t_{i+n+1}-t_i)\frac{1}{\prod_{\substack{l=0\\l\neq j+1}}^{n+1}(t_{i+j+1}-t_{i+l})} + \frac{t_{i+n+2}-x}{t_{i+n+2}-t_{i+1}}(t_{i+n+2}-t_{i+1})\frac{1}{\prod_{\substack{l=0\\l\neq j}}^{n+1}(t_{i+j+1}-t_{i+l+1})}$$

$$= \frac{1}{\prod_{\substack{l=0\\l\neq j}}^{n}(t_{i+j+1}-t_{i+l+1})}\left[\frac{x-t_i}{t_{i+j+1}-t_i} + \frac{t_{i+n+2}-x}{t_{i+j+1}-t_{i+n+2}}\right]$$

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$$= \frac{1}{\prod_{\substack{l\neq 0\\ l\neq j}}^{n} (t_{i+j+1} - t_{i+l+1})} \left[\frac{x(t_i - t_{i+n+2}) + t_{i+j+1}(t_{i+n+2} - t_i)}{(t_{i+j+1} - t_i)(t_{i+j+1} - t_{i+n+2})} \right]$$

$$= \frac{t_{i+n+2} - t_i}{\prod_{\substack{l=0\\ l\neq j}}^{n+1} (t_{i+j+1} - t_{i+l+1})} (t_{i+j+1} - x)$$

$$\therefore B_{i,n+1} = (t_{i+n+2} - t_i) \sum_{j=0}^{n+2} \frac{(t_{i+j} - x)_+^{n+1}}{\prod_{\substack{l=0\\ l\neq j}}^{n+2} (t_{i+j} - t_{i+l})}$$

$$B_{-i,d} = (t_{-i+d+1} - t_{-i}) \sum_{j=0}^{d+1} \frac{(t_{-i+j} - x)_+^d}{\prod_{\substack{l=0\\ l\neq j}}^{d+1} (t_{-i+j} - t_{-i+l})}$$

$$= (t_{g+1+d+1-i} - t_{g+1-i}) \sum_{j=0}^{d+1} \frac{(t_{g+1+j-i} - (x+b-a))_+^d}{\prod_{\substack{l=0\\ l\neq j}}^{d+1} (t_{g+1+j-i} - t_{g+1+l-i})}$$

$$= B_{g+1-i,d}(x+b-a)$$

let $s(x) = \sum_{i=-d}^{g} c_i B_{i,d}(x)$

$\because s(a) = s(b)$

$\therefore c_{-d} B_{-d,d} + \cdots + c_0 B_{0,d} = c_{g+1} B_{g+1,d}(b) + \cdots + c_{g+1-d} B_{g+1-d,d}(b)$

$\therefore (c_0 - c_{g+1}) B_{0,d}(a) + (c_{-1} - c_g) B_{-1,d}(a) + \cdots + (c_{-d} - c_{g+1-d}) B_{-d,d}(a) = 0$

$\therefore c_0 = c_{g+1}, \; c_{-1} = c_g, \; c_{-2} = c_{g-1} \cdots \; c_{-d} = c_{g+1-d}$

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$$B'_{i,d} = (d+1)\left(\frac{B_{i,d-1}}{t_{i+d}-t_i} - \frac{B_{i+1,d-1}}{t_{i+d+1}-t_{i+1}}\right) \to \text{if } B_{i,d-1} \in C^{d-1-k} \text{ implies } B_{i,d} \in C^{d-k}$$

proof:

when $d=0$

$$B_{i,0} = 1 \qquad x \in [t_i, t_{i+1})$$

$$B'_{i,0} = 0$$

when $d=1$

$$B_{i,1} = \frac{x-t_i}{t_{i+1}-t_i} B_{i,0} + \frac{t_{i+2}-x}{t_{i+2}-t_{i+1}} B_{i+1,0}$$

$$B'_{i,1} = \frac{B_{i,0}}{t_{i+1}-t_i} - \frac{B_{i+1,0}}{t_{i+2}-t_{i+1}}$$

when $d=2$

$$B_{i,2} = \frac{x-t_i}{t_{i+2}-t_i} B_{i,1} + \frac{t_{i+3}-x}{t_{i+3}-t_{i+1}} B_{i+1,1}$$

$$B'_{i,2} = \frac{B_{i,1}}{t_{i+2}-t_i} + \frac{x-t_i}{t_{i+2}-t_i} B'_{i,1} + \frac{-1}{t_{i+3}-t_{i+1}} B_{i+1,1} + \frac{t_{i+3}-x}{t_{i+3}-t_{i+1}} B'_{i+1,1}$$

$$= \frac{B_{i,1}}{t_{i+2}-t_i} + \frac{x-t_i}{t_{i+2}-t_i}\left(\frac{B_{i,0}}{t_{i+1}-t_i} - \frac{B_{i+1,0}}{t_{i+2}-t_{i+1}}\right) - \frac{B_{i+1,1}}{t_{i+3}-t_{i+1}}$$

$$+ \frac{t_{i+3}-x}{t_{i+3}-t_{i+1}}\left(\frac{B_{i+1,0}}{t_{i+2}-t_{i+1}} - \frac{B_{i+2,0}}{t_{i+3}-t_{i+2}}\right)$$

$$= \frac{B_{i,1}}{t_{i+2}-t_i} + \frac{1}{t_{i+2}-t_i}\left(\frac{x-t_i}{t_{i+1}-t_i} B_{i,0} + \frac{t_{i+2}-x}{t_{i+2}-t_{i+1}} B_{i+1,0}\right) - \frac{B_{i+1,0}}{t_{i+2}-t_{i+1}}$$

$$-\frac{B_{i+1,1}}{t_{i+3}-t_{i+1}} + \frac{1}{t_{i+3}-t_{i+1}}\left(-\frac{x-t_{i+3}}{t_{i+2}-t_{i+1}}B_{i+1,0} - \frac{t_{i+3}-x}{t_{i+3}-t_{i+2}}B_{i+2,0}\right)$$

$$= 2\frac{B_{i,1}}{t_{i+2}-t_i} - \frac{B_{i+1,1}}{t_{i+3}-t_{i+1}} + \frac{1}{t_{i+3}-t_{i+1}}\left(-\frac{x-t_{i+3}+t_{i+3}-t_{i+1}}{t_{i+2}-t_{i+1}}B_{i+1,0} - \frac{t_{i+3}-x}{t_{i+3}-t_{i+2}}B_{i+2,0}\right)$$

$$= 2\left[\frac{B_{i,1}}{t_{i+2}-t_i} - \frac{B_{i+1,1}}{t_{i+3}-t_{i+1}}\right]$$

when $d = n$, show that $(n+1)$ is true

$$B_{i,n+1} = \frac{x-t_i}{t_{i+n+1}-t_i}B_{i,n} + \frac{t_{i+n+2}-x}{t_{i+n+2}-t_{i+1}}B_{i+1,n}$$

$$B'_{i,n+1} = \frac{B_{i,n}}{t_{i+n+1}-t_i} - \frac{B_{i+1,n}}{t_{i+n+2}-t_{i+1}} + \frac{x-t_i}{t_{i+n+1}-t_i}(d)\left(\frac{B_{i,n-1}}{t_{i+n}-t_i} - \frac{B_{i+1,n-1}}{t_{i+n+1}-t_{i+1}}\right)$$
$$+ \frac{t_{i+n+2}-x}{t_{i+n+2}-t_{i+1}}(d)\left(\frac{B_{i+1,n-1}}{t_{i+n+1}-t_{i+1}} - \frac{B_{i+2,n-1}}{t_{i+n+2}-t_{i+2}}\right)$$

$$= \frac{B_{i,n}}{t_{i+n+1}-t_i} - \frac{B_{i+1,n}}{t_{i+n+2}-t_{i+1}} + \frac{d}{t_{i+n+1}-t_i}\left(\frac{x-t_i}{t_{i+n}-t_i}B_{i,n-1} + \frac{t_{i+n+1}-x}{t_{i+n+1}-t_{i+1}}B_{i+1,n-1}\right)$$
$$-(d)\frac{B_{i+1,n-1}}{t_{i+n+1}-t_{i+1}} + \frac{d}{t_{i+n+2}-t_{i+1}}\frac{t_{i+n+2}-x}{t_{i+n+1}-t_{i+1}}B_{i+1,n-1}$$
$$-\frac{d}{t_{i+n+2}-t_{i+1}}\frac{t_{i+n+2}-x}{t_{i+n+2}-t_{i+2}}B_{i+2,n-1}$$

$$= (d+1)\frac{B_{i,n}}{t_{i+n+1}-t_i} - \frac{B_{i+1,n}}{t_{i+n+1}-t_{i+1}} - \frac{d}{t_{i+n+2}-t_{i+1}}\left(\frac{x-t_{i+1}}{t_{i+n+1}-t_{i+1}}B_{i+1,n-1} + \frac{t_{i+n+2}-x}{t_{i+n+2}-t_{i+2}}B_{i+2,n-1}\right)$$

$$= \left(\frac{B_{i,n}}{t_{i+n+1}-t_i} - \frac{B_{i+1,n}}{t_{i+n+2}-t_{i+1}}\right)(d+1)$$

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$$\therefore B''_{i,d} = (d-1)(d)\left(\frac{1}{t_{i+d}-t_i}\right)\left(\frac{B_{i,d-2}}{t_{i+d-1}-t_i} - \frac{B_{i+1,d-2}}{t_{i+d}-t_{i+1}}\right)$$

$$-(d-1)(d)\left(\frac{1}{t_{i+d+1}-t_{i+1}}\right)\left(\frac{B_{i+1,d-2}}{t_{i+d}-t_{i+1}} - \frac{B_{i+2,d-2}}{t_{i+d+1}-t_{i+2}}\right).$$

$$=(d-1)(d)\left[\frac{B_{i,d-2}}{(t_{i+d}-t_i)(t_{i+d-1}-t_i)}\right.$$

$$\left.+\frac{B_{i+1,d-2}}{t_{i+d}-t_{i+1}}\,\frac{t_{i+1}-t_{i+d+1}+t_i-t_{i+d}}{(t_{i+d}-t_i)(t_{i+d+1}-t_{i+1})} + \frac{B_{i+2,d-2}}{(t_{i+d+1}-t_{i+1})(t_{i+d+1}-t_{i+2})}\right]$$

$$s_x(x) = \sum_{i=-d}^{g} c_i^x B_{i,d}(x) \qquad s_x' = \sum_{i=-d}^{g} c_i^x B_{i,d}'$$

$$s_y(y) = \sum_{i=-d}^{g} c_i^y B_{i,d}(y) \qquad s_y' = \sum_{i=-d}^{g} c_i^y B_{i,d}'$$

$$k = \frac{s_x' s_y'' - s_y' s_x''}{\sqrt{s_x'^2 + s_y'^2}^{\,3}}$$

Recurrence Relation:

$$\left(\frac{B_{i,d-1}}{t_{i+d}-t_i},\ \frac{-B_{i+1,d-1}}{t_{i+d+1}-t_{i+1}}\right)$$

$$\left(\frac{1}{t_{i+d}-t_i}\,\frac{B_{i,d-2}}{t_{i+d-1}-t_i},\ \frac{1}{t_{i+d}-t_i}\,\frac{-B_{i+1,d-2}}{t_{i+d}-t_{i+1}} + \frac{-1}{t_{i+d+1}-t_{i+1}}\,\frac{B_{i+1,d-2}}{t_{i+d}-t_{i+1}},\right.$$

$$\left.\frac{-1}{t_{i+d+1}-t_{i+1}}\,\frac{-B_{i+2,d-2}}{t_{i+d+1}-t_{i+2}}\right)$$

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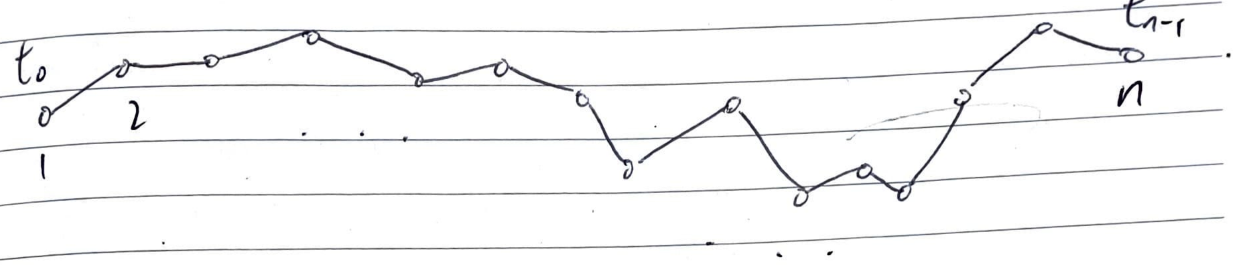
B-Spline Interpolation

let d be degree, odd number



$$\text{knot} = (\underbrace{t_{-d}, \dots, t_0}_{\text{boundary knots}}, \dots, \underbrace{t_{n-1}, \dots, t_{n-1+d}}_{\text{boundary knots}})$$

$$t_{-d} = t_{-d+1} = \dots = t_0$$

$$t_{n-1} = t_n = \dots = t_{n-1+d}$$

$$s = \sum_{i=-d}^{n-2} c_i B_{i,d}$$

no. of unkown variables $= n-2+d+1 = n-1+d$

no. of data points $=$ no. of equations $= n$

remaining $d-1$ equations can be obtained from natural end boundary conditions, i.e.

$$s''(t_0) = s''''(t_0) = \dots = 0$$

$$s''(t_{n-1}) = s''''(t_{n-1}) = \dots = 0$$

Moreover, $c_{-d} = x_1$, $c_{n-2} = x_n$

No.

Date. 21 · 6 · 2017

B-Spline Interpolation

Ex.

$d = 3$, $n = 8$

$t_{-3} = t_{-2} = t_{-1} = t_0$

$B_{0,0} = 1 \quad B_{0,1} = 0 \quad B_{0,2} = 0 \quad B_{0,3} = 0$

$B_{-1,0} = 0 \quad B_{-1,1} = 1 \quad B_{-1,2} = 0 \quad B_{-1,3} = 0$

$B_{-2,0} = 0 \quad B_{-2,1} = 0 \quad B_{-2,2} = 1 \quad B_{-2,3} = 0$

$B_{-3,0} = 0 \quad B_{-3,1} = 0 \quad B_{-3,2} = 0 \quad B_{-3,3} = 1$

$$B_{-1,1} = \frac{x - t_0}{t_0 - t_{-1}} B_{-1,0} + \frac{t_1 - x}{t_1 - t_0} B_{0,0}$$

$$= 1$$

$B'_{0,0} = 0 \quad B'_{0,1} = \frac{1}{t_1 - t_0} \quad B'_{0,2} = 0 \quad B'_{0,3} = 0$

$B'_{-1,0} = 0 \quad B'_{-1,1} = \frac{-1}{t_1 - t_0} \quad B'_{-1,2} = \frac{2}{t_1 - t_{-1}} \quad B'_{-1,3} = 0$

$B'_{-2,0} = 0 \quad B'_{-2,1} = 0 \quad B'_{-2,2} = \frac{-2}{t_1 - t_{-1}} \quad B'_{-2,3} = \frac{3}{t_1 - t_{-2}}$

$B'_{-3,0} = 0 \quad B'_{-3,1} = 0 \quad B'_{-3,2} = 0 \quad B'_{-3,3} = \frac{-3}{t_1 - t_{-2}}$

$$B'_{-2,3} = 3\left(\frac{B_{-2,2}}{t_1 - t_{-2}} - \frac{B_{-1,2}}{t_2 - t_{-1}}\right)$$

$$B'_{-3,3} = 3\left(\frac{B_{-3,2}}{t_0 - t_{-3}} - \frac{B_{-2,2}}{t_1 - t_{-2}}\right)$$

$$B'_{-2,2} = 2\left(\frac{B_{-2,1}}{t_0 - t_{-2}} - \frac{B_{-1,1}}{t_1 - t_{-1}}\right)$$

$$B''_{-1,3} = 3\left(\frac{B'_{-1,2}}{t_2 - t_{-1}} - \frac{B'_{0,2}}{t_3 - t_0}\right)$$

$$= 6 \frac{1}{t_2 - t_{-1}} \frac{1}{t_1 - t_{-1}}$$

$$B''_{-2,3} = 3\left(\frac{B'_{-2,2}}{t_1 - t_{-2}} - \frac{B'_{-1,2}}{t_2 - t_{-1}}\right)$$

$$= -6\left(\frac{1}{t_1 - t_{-2}} \frac{1}{t_1 - t_{-1}} + \frac{1}{t_2 - t_{-1}} \frac{1}{t_1 - t_{-1}}\right)$$

$$B''_{-3,3} = 3\left(\frac{B'_{-3,2}}{t_0 - t_{-3}} - \frac{B'_{-2,2}}{t_1 - t_{-2}}\right)$$

$$= 6 \frac{1}{t_1 - t_{-2}} \frac{1}{t_1 - t_{-1}}$$